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Abstract. We consider a functional on the Wiener space which is smooth and not degenerated in Malliavin sense and we give a criterion for the strict positivity of the density, that we can use to state lower bounds as well. The results are based on the representation of the density in terms of the Riesz transform introduced in Malliavin and Thalmaier [16] and on the estimates of the Riesz transform given in Bally and Caramellino [3].

Keywords: Riesz transform, Malliavin calculus, strict positivity and lower bounds for the density.

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1 Introduction

The aim of this paper is to study the strict positivity and lower bounds for the density of a functional on the Wiener space. Although the two problems are related each other, the hypothesis under which the results may be obtained are different. Just to make clear what we expect to be these hypothesis, consider the example of a d dimensional diffusion process X_t solution of $dX_t = \sum_{j=1}^m \sigma_j(X_t) \circ dW_t^j + b(X_t)dt$ where $\circ dW_t^j$ denotes the Stratonovich integral. The skeleton associated to this diffusion process is the solution $x_t(\phi)$ of the equation $dx_t(\phi) = \sum_{j=1}^m \sigma_j(x_t(\phi))\phi_t^j dt + b(x_t(\phi))dt$, for a square integrable ϕ. The celebrated support theorem of Stroock and Varadhan guarantees that the support of the law of X_t is the closure of the set of points x which are attainable by a skeleton, that is $x = x_t(\phi)$ for some control $\phi \in L^2([0,T])$. Suppose now that the law of X_t has a continuous density p_{X_t} with

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respect to the Lebesgue measure. Then in order to get a criterion for $p_{X_t}(x) > 0$, we prove that this holds if x is attainable, that is $x = x_t(\phi)$ for some ϕ, and a suitable non degeneracy assumption holds in x. The second problem is to give a lower bound for $p_{X_t}(x)$ and this can be achieved if a non degeneracy condition holds all along the curve $x(\phi)$ which arrives in x at time t. Roughly speaking, the idea is the following. If one has a non degeneracy condition all along the skeleton curve arriving in x at time t, one may give a lower bound for the probability to remain in the tube up to $t - \delta$ for a small $\delta > 0$ and then one employs an argument based on Malliavin calculus in order to focus on the point x - essentially this means that one is able to give a precise estimate of the behavior of the diffusion in short time (between $t - \delta$ and t). This allows one to obtain a lower bound for $p_{X_t}(x)$. If one is not interested in lower bounds but only in the strict positivity property, the argument is the same but one does not need to estimate the probability to remain in the tube: using the support theorem one knows that this probability is strictly positive (but this is just qualitative, so one has no lower bound for it) and then one focuses on the point x using again the same argument concerning the behavior of the diffusion in short time. So one needs the non degeneracy condition in x only.

The two problems mentioned above have been intensively studied in the literature. Let us begin with the strict positivity. At the best of our knowledge the first probabilistic approach to this problem is due to Ben-Arous and Leandre [8], who used Malliavin calculus in order to give necessary and sufficient conditions allowing one to have $p_{X_t}(x) > 0$ for a diffusion process (as above). They proved that if Hörmander's condition holds then $p_{X_t}(x) > 0$ if and only if x is attainable by a skeleton $x_t(\phi)$ such that $\psi \mapsto x_t(\psi)$ is a submersion in ϕ. The argument they used is based on the inverse function theorem and on a Girsanov transformation. All the papers which followed developed in some way their techniques. First, Aida, Kusuoka and Stroock [1] gave a generalization of this criterion in an abstract framework which still permits to exhibit a notion of skeleton. Then Hirsch and Song [11] studied a variant of such a criterion for a general functional on the Wiener space using capacities and finally Leandre [14] obtained similar results for diffusion processes on manifolds. Notice that once one has a criterion of the above type there is still a non trivial problem to be solved: one has to exhibit the skeleton which verifies the submersion property. So, number of authors dealt with concrete examples in which they are able to use in a more or less direct way the argument of Ben-Arous and Leandre: Bally and Pardoux [7] dealt with parabolic stochastic heat equations, Millet and Sanz-Solé [18] worked with hyperbolic stochastic partial differential equations, Fournier [10] considered jump type equations, Dalang and D. Nualart [9] used such positivity results for building a potential theory for SPDE's and E. Nualart [20] has recently proved results in this direction again for solutions of SPDE's.

Concerning lower bounds for the density, a first result was found by Kusuoka and Stroock [13] for diffusion processes that verify a strong uniform Hörmander condition. Afterwards Kohatsu-Higa [12] obtained lower bounds for general functionals on the Wiener space under a uniform ellipticity condition and Bally [2] proved results under local ellipticity conditions. Recently, Gaussian type lower and upper bounds are studied in E. Nualart and Quer-Sardanyons [21] for the nonlinear stochastic heat equation.

The present paper gives a contribution in this framework: we study the strict positivity and lower bounds for the density of a general functional on the Wiener space starting from a result (Proposition 3.1) which gives the behavior of a small perturbation of a Gaussian random variable - it corresponds to the study of a diffusion process in short time (between $t-\delta$ and t). This is a consequence of an abstract result (Theorem 2.4) in which the distance between the local density functions of two random variables (doesn't matter if one of them is Gaussian) is studied. It is worth to stress that Theorem 2.4 is of interest in itself and can be linked to the implicit function theorem in order to get further estimates which can be used to handle the same problem under Hörmander type conditions (see [4]).
So, our main result (see Theorem 3.3) gives sufficient conditions in order to obtain the following lower bound for the law of F around a point $y \in \mathbb{R}^d$: there exists $\eta > 0$ and $c(y) > 0$ such that

$$\mathbb{P}(F \in A) \geq c(y)\mathrm{Leb}_d(A) \quad \text{for every Borel set } A \subset B_\eta(y),$$

Leb_d denoting the Lebesgue measure on $\mathbb{R}^d$. In particular, if the law of F is absolutely continuous on $B_\eta(y)$ then the density p_F satisfies $p_F(x) \geq c(y) > 0$ for every $x \in B_\eta(y)$. Essentially, our conditions are that y belongs to the support of the law of F and an ellipticity-type condition holds around y.

In our examples, we first deal with an Ito process X_t defined as a component of a diffusion process, that is

$$\begin{aligned} X_t =& x_0 + \sum_{j=1}^m \int_0^t \sigma_j(X_t, Y_t) dW_t^j + \int_0^t b(X_t, Y_t) dt \\ Y_t =& y_0 + \sum_{j=1}^m \int_0^t \alpha_j(X_t, Y_t) dW_t^j + \int_0^t \beta(X_t, Y_t) dt. \end{aligned}$$

Notice that for diffusion processes, we get an example which is essentially the same treated in Ben Arous and Leandre [8] and in Aida, Kusouka and Stroock [1]. Let $(x(\phi), y(\phi))$ denote the skeleton associated to the diffusion pair (X, Y) and let $x = x_t(\phi)$ for some suitable control ϕ. Then, whenever a continuous local density p_{X_t} of X_t exists in x, we prove that if $\sigma\sigma^*(x, y_t(\phi)) > 0$ then $p_{X_t}(x) > 0$. And moreover, if $\inf_{s\leq t}\inf_y \sigma\sigma^*(x_s(\phi), y) \geq \lambda_* > 0$ and $x_s(\phi)$ belongs to a suitable class of paths (see Theorem 4.1 for details), then a lower bound for $p_{X_t}(x)$ can be written in terms of the lower estimates for the probability that Ito processes remain near a path proved in Bally, Fernández and Meda in [6].
As a second example, in Section 4.2 we treat the two dimensional diffusion process

$$dX_t^1 = \sigma_1(X_t) dW_t + b_1(X_t) dt, \quad dX_t^2 = b_2(X_t) dt$$

which is degenerated in any point $x \in \mathbb{R}^2$. We assume that x is attainable by a skeleton $x_t(\phi)$ and that $|\sigma_1(x)| > 0$ and $|\partial_1 b_2(x)| > 0$ - which amounts to say that the weak Hörmander condition holds in the point x. We prove that under this hypothesis one has $p_{X_t}(x) > 0$. For this example Bally and Kohatsu-Higa [5] have already given a lower bound for the density under the stronger hypothesis that $\inf_{s\leq t} |\sigma(x_s(\phi))| > 0$ and $\inf_{s\leq t} |\partial_1 b_2(x_s(\phi))| > 0$. So the same non degeneracy condition holds but along the whole curve $x_s(\phi), 0 \leq s \leq t$. Notice that we use a skeleton $x_s(\phi)$ which arrives in x but we do not ask for the immersion property (according the result of Ben-Arous and Leandre it follows that a skeleton which verifies the immersion property exists also, but we do not know how to produce it directly and we do not need it). And it seems clear to us that our criterion may be used for SPDE's as well and would simplify the proofs given in the already mentioned papers.

The paper is organized as follows. In Section 2 we first state localized representation formulas for the density by means of the Riesz transform (see Section 2.1) and then we study the distance between the local densities of two random variables (see Section 2.2). Section 3 is devoted to the results on the perturbation of a Gaussian random variable (see Section 3.1) and to the study of the strict positivity and the lower bounds for the density of a general functional on the Wiener space (see Section 3.2). We finally discuss our examples in Section 4.

2 Localized integration by parts formulas

We consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with an infinite dimensional Brownian motion $W = (W^n)_{n\in\mathbb{N}}$ and we use the Malliavin calculus in order to obtain integration by parts formulas. We refer to D. Nualart [19] for notation and basic results. We denote by $\mathbb{D}^{k,p}$ the space of the random variables which are k times differentiable in Malliavin sense in L^p and for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$ we denote by $D^\alpha F$ the Malliavin derivative of F corresponding to the multi-index α. So, $\mathbb{D}^{m,p}$ is the closure of the space of the simple functionals with respect to the Malliavin Sobolev norm

$$\|F\|_{m,p}^p = \|F\|_p^p + \sum_{k=1}^m \mathbb{E}\big(|D^{(k)}F|^p\big)$$

where

$$|D^{(k)}F|^2 = \sum_{|\alpha|=k} \int_{[0,\infty)^k} \left|D^\alpha_{s_1,\ldots,s_k}F\right|^2 \, ds_1, \ldots ds_k.$$

In the special case $k = 1$, we consider the notation

$$|DF|^2 := |D^{(1)}F|^2 = \sum_{\ell=0}^\infty \int_{[0,\infty)} \left|D^\ell_s F\right|^2 \, ds,$$

(for the sake of clearness, we recall that D^ℓ stands for the Malliavin derivative w.r.t. W^ℓ - and not the derivative of order ℓ). Moreover, for $F = (F^1, \ldots, F^d), F^i \in \mathbb{D}^{1,2}$, we let σ_F denote the Malliavin covariance matrix associated to F :

$$\sigma_F^{i,j} = \langle DF^i, DF^j \rangle = \sum_{k=1}^{\infty} \int_0^{\infty} D_s^k F^i D_s^k F^j ds, \quad i,j = 1, \ldots, d.$$

If σ_F is invertible, we denote through $\widehat{\sigma}_F$ the inverse matrix. Finally, as usual, the notation L will be used for the OrnsteinUhlenbeck operator.

2.1 Localized representation formulas for the density

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Consider now an integrable random variable U taking values on $[0,1]$ and set

$$d\mathbb{P}_U = U d\mathbb{P}.$$

$\mathbb{P}_U$ is a non negative measure (but generally not a probability measure) and we set $\mathbb{E}_U$ the expectation (integral) w.r.t. $\mathbb{P}_U$. For $F \in \mathbb{D}^{k,p}$, we define

$$\|F\|_{p,U}^p = \mathbb{E}_U(|F|^p) \quad \text{and} \quad \|F\|_{k,p,U}^p = \|F\|_{p,U}^p + \sum_{i=1}^k \mathbb{E}_U(|D^{(i)} F|^p).$$

We assume that $U \in \mathbb{D}^{1,\infty}$ and we consider the following condition:

$$m_U(p) := 1 + \mathbb{E}_U(|D \ln U|^p) < \infty, \quad \text{for every } p \in \mathbb{N}. \tag{2.1}$$

Mall1

(2.1) could seem problematic because U may vanish and then $D(\ln U)$ is not well defined. Nevertheless we make the convention that $D(\ln U) = \frac{1}{U} DU \,\mathbf{1}_{\{U \neq 0\}}$ (in fact this is the quantity we are really concerned in). Since $U > 0$ $\mathbb{P}_U$-a.s. and DU is well defined, the relation $\|\ln U\|_{1,p,U} < \infty$ makes sense.
We give now the integration by parts formula with respect to $\mathbb{P}_U$ (that is, locally) and we study some consequences concerning the regularity of the law starting from the results in Bally and Caramellino [3] (see also Shigekawa [22] or Malliavin [15]). In particular, for $F \in (\mathbb{D}^{1,\infty})^d$, we will need that the Malliavin covariance matrix σ_F is invertible a.s. under $\mathbb{P}_U$, so we call again $\widehat{\sigma}_F$ the inverse of σ_F on the set $\{U \neq 0\}$.
Let Q_d denote the Poisson kernel on $\mathbb{R}^d$: Q_d is the fundamental solution of the equation $\Delta Q_d = \delta_0$ in $\mathbb{R}^d$ (δ_0 denoting the Dirac mass at the origin) and is given by

$$Q_1(x) = \max(x, 0), \quad Q_2(x) = \mathcal{A}_2^{-1} \ln |x| \quad \text{and} \quad Q_d(x) = -\mathcal{A}_d^{-1} |x|^{2-d}, d > 2, \tag{2.2}$$

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where for $d \geq 2$, $\mathcal{A}_d$ is the area of the unit sphere in $\mathbb{R}^d$. Then one has

1 **Lemma 2.1.** *Assume that (2.1) holds. Let $F = (F_1, ..., F_d)$ be such that $F_i \in \mathbb{D}^{2,\infty}$, $i = 1, \ldots, d$. Assume that $\det \sigma_F > 0$ on the set $\{U \neq 0\}$ and moreover*

$$\mathbb{E}_U((\det \sigma_F)^{-p}) < \infty \quad \forall p \in \mathbb{N}. \tag{2.3}$$

Law1

Let $\widehat{\sigma}_F$ be the inverse of σ_F on the set $\{U \neq 0\}$. Then the following statements hold.

A. *For every* $f \in C_b^\infty(R^d)$ *and* $V \in \mathbb{D}^{1,\infty}$ *one has*

$$\mathbb{E}_U(\partial_i f(F)\, V) = \mathbb{E}_U(f(F) H_{i,U}(F,V)), \quad i = 1, \dots, d, \textit{with}$$

$$H_{i,U}(F,V) = \sum_{j=1}^d \Big(V \widehat{\sigma}_F^{ji} LF^j - \big\langle D(V\widehat{\sigma}_F^{ji}), DF^j \big\rangle - V\widehat{\sigma}_F^{ji} \big\langle D\ln U, DF^j \big\rangle \Big). \tag{2.4}$$

B. *Let* Q_d *be the Poisson kernel in* $\mathbb{R}^d$ *given in* (2.2). *Then for every* $p > d$ *one has*

$$\mathbb{E}_U(|\nabla Q_d(F-x)|^{\frac{p}{p-1}})^{\frac{p-1}{p}} \le C_{p,d} \mathbb{E}_U(|H_U(F,1)|^p)^{k_{p,d}} \tag{2.5}$$

where $C_{p,d}$ *is a universal constant depending on* p *and* d *and* $k_{p,d} = (d-1)/(1-d/p)$.
C. *Under* $\mathbb{P}_U$, *the law of* F *is absolutely continuous and has a continuous density* $p_{F,U}$ *which may be represented as*

$$p_{F,U}(x) = \sum_{i=1}^d \mathbb{E}_U(\partial_i Q_d(F-x) H_{i,U}(F,1)). \tag{2.6}$$

Moreover, there exist constants $C > 0$ *and* $p, q > 1$ *depending on* d *only such that*

$$p_{F,U}(x) \le C \gamma_{F,U}(p)^q n_{F,U}(p)^q m_U(p)^q \tag{2.7}$$

with $m_U(p)$ *given in* (2.1),

$$\gamma_{F,U}(p) = 1 + \mathbb{E}_U(|\det \sigma_F|^{-p}) \quad \textit{and} \quad n_{F,U}(p) = 1 + \|F\|_{2,p,U} + \|LF\|_{p,U}. \tag{2.8}$$

Finally, if $V \in \mathbb{D}^{1,\infty}$ *then there exist* $C > 0$ *and* $p, q > 1$ *depending on* d *such that*

$$p_{F,UV}(x) \le C \gamma_{F,U}(p)^q n_{F,U}(p)^q m_U(p)^q \|V\|_{1,p,U}. \tag{2.9}$$

Proof. **A.** The standard integration by parts formula in Malliavin calculus gives (vector notations)

$$\mathbb{E}_U(\nabla f(F) V) = \mathbb{E}(\nabla f(F) UV) = \mathbb{E}(f(F) H(F, UV))$$

where, setting $DU = U \times D(\ln U)$, one has

$$\begin{aligned} H(F,UV) &= VU\widehat{\sigma}_F LF - \langle D(VU\widehat{\sigma}_F), DF \rangle \\ &= U(V\widehat{\sigma}_F LF - \langle D(V\widehat{\sigma}_F), DF \rangle) - V\widehat{\sigma}_F \langle D\ln U, DF \rangle), \end{aligned}$$

So, $H(F,UV) = UH_U(F,V)$, and (2.4) is proved.
B. This point straightforwardly follows from the results and the techniques in Bally and Caramellino [3].
C. (2.6) again follows from [3], while (2.7) is a consequence of the inequality

$$\|H_U(F,V)\|_{p,U} \le C \gamma_{F,U}(p)^q n_{F,U}(p)^q m_U(p) \|V\|_{1,p,U}, \tag{2.10}$$

holding for suitable $C>0$ and $p,q>1$ depending on d only. This can be proved by applying the Hölder inequality to (2.4) (further details can be found in the proof of next Proposition 2.2). So, by using the Hölder inequality to (2.6) and by considering both (2.5) and (2.10), one gets (2.7). Finally, in order to prove (2.9) we formally write (the rigorous arguments can be found in [3])

$$\begin{aligned} p_{F,UV}(x) =&\mathbb{E}_{UV}(\delta_0(F-x)) = \mathbb{E}_{UV}(\triangle Q_d(F-x)) = \mathbb{E}_U(\triangle Q_d(F-x)V)\\ =&\mathbb{E}_U(\langle \nabla Q_d(F-x), H_U(F,V)\rangle). \end{aligned}$$

Then using (2.5) and(2.10) one obtains (2.9). $\square$

2.2 The distance between two density functions

We compare now the densities of the laws of two random variables under $\mathbb{P}_U$.

Proposition 2.2. *Assume that (2.1) holds. Let $F=(F_1,...,F_d)$ and $G=(G_1,...,G_d)$ be such that $F_i,G_i\in\mathbb{D}^{2,\infty}$, $i=1,\dots,d$, and*

$$\gamma_{F,G,U}(p):=1+\sup_{0\leq\varepsilon\leq 1}\mathbb{E}_U((\det\sigma_{G+\varepsilon(F-G)})^{-p}))<\infty,\quad \forall p\in\mathbb{N}.$$

Then under $\mathbb{P}_U$ the laws of F and G are absolutely continuous with respect to the Lebesgue measure with continuous densities $p_{F,U}$ and $p_{G,U}$ respectively. Moreover, there exist a constant $C>0$ and two integers $p,q>1$ depending on d only such that

$$|p_{F,U}(y)-p_{G,U}(y)|\leq C\,\gamma_{F,G,U}(p)^q n_{F,G,U}(p)^q m_U(p)^q\,\|\Delta_2(F,G)\|_{p,U} \tag{2.11}$$

with $m_U(p)$ given in (2.1) and

$$\begin{aligned} &\Delta_2(F,G)=|D(F-G)|+\left|D^{(2)}(F-G)\right|+|L(F-G)|\,,\\ &n_{F,G,U}(p)=1+\|F\|_{2,p,U}+\|G\|_{2,p,U}+\|LF\|_{p,U}+\|LG\|_{p,U}. \end{aligned} \tag{2.12}$$

Moreover, since $|U|\leq 1$ almost surely, using Meyer's inequality one has

$$|p_{F,U}(y)-p_{G,U}(y)|\leq C\gamma_{F,G,U}(p)^q m_U(p)^q(1+\|F\|_{2,p}+\|G\|_{2,p})^q\,\|F-G\|_{2,p}\,. \tag{2.13}$$

Proof. Throughout this proof, C,p,q (that can vary from line to line) will be universal constants depending on d only

By applying Lemma 2.1, we first notice that under $\mathbb{P}_U$ the laws of F and G are both absolutely continuous with respect to the Lebesgue measure and the densities can be written as

$$\begin{aligned} p_{F,U}(y)&=\mathbb{E}_U(\langle\nabla Q_d(F-y),H_U(F,1)\rangle) \text{ and}\\ p_{G,U}(y)&=\mathbb{E}_U(\langle\nabla Q_d(G-y),H_U(G,1)\rangle). \end{aligned} \tag{2.14}$$

Step 1. We prove that for $V\in\mathbb{D}^{1,\infty}$, on the set $\{U\neq 0\}$ one has

$$|H_U(F,V)-H_U(G,V)|\leq C\,A_{F,G}\,B_{F,G}\,(1+|D\ln U|)(|V|+|DV|)\times\Delta_2(F,G) \tag{2.15}$$

where on the set $\{U \neq 0\}$ (that is, where the inverse Malliavin covariance matrices $\widehat{\sigma}_F$ and $\widehat{\sigma}_G$ are actually well defined) the above quantities are equal to

$$\begin{aligned} A_{F,G} &= (1 \vee \det \widehat{\sigma}_F)^2 (1 \vee \det \widehat{\sigma}_G)^2, \\ B_{F,G} &= (1 + |DF| + |DG| + \left|D^{(2)}F\right| + \left|D^{(2)}G\right|)^{d(d-1)} (1 + |LF| + |LG|). \end{aligned}$$

So, we work on the set $\{U \neq 0\}$. We first notice that

$$\begin{aligned} \left|\widehat{\sigma}_F^{i,j} - \widehat{\sigma}_G^{i,j}\right| \leq & C(1 \vee \det \widehat{\sigma}_F)(1 \vee \det \widehat{\sigma}_G) \left|D(F-G)\right| (|DF| + |DG|)^{d(d-1)}, \\ \left|D\widehat{\sigma}_F^{i,j} - D\widehat{\sigma}_G^{i,j}\right| \leq & C(1 \vee \det \widehat{\sigma}_F)^2 (1 \vee \det \widehat{\sigma}_G)^2 (|D(F-G)| + \left|D^{(2)}(F-G)\right|) \\ & \times (|DF| + |DG| + \left|D^{(2)}F\right| + \left|D^{(2)}G\right|)^{d(d-1)}. \end{aligned}$$

Then a straightforward computation gives (2.15). Now, using (2.15) and the Hölder inequality one has

$$\|H_U(F,V) - H_U(G,V)\|_{p,U} \leq C n_{F,G,U}(p')^q m_U(p') \|V\|_{1,p',U} \|\Delta_2(F,G)\|_{p',U} \tag{2.16}$$ Mall6

Step 2. By using arguments similar to the ones developed in Step 1, we get

$$\|H_U(F,V)\|_{p,U} \leq C\,\gamma_{F,U}(p')^q n_{F,U}(p')^q m_U(p') \|V\|_{1,p',U}, \tag{2.17}$$ Bis4

$n_{F,U}(p')$ and $\gamma_{F,U}(p')$ being defined in (2.8). So, by taking $p = (d+1)/d$ in (2.5) and by using (2.17) with $V = 1$ one gets

$$\|\nabla Q_d(F-y)\|_{p/(p-1),U} \leq C\,\gamma_{F,U}(p')^q\, n_{F,U}(p')^q m_U(p')^q. \tag{2.18}$$ Mall7

Step 3. By using (2.14), we can write

$$\begin{aligned} p_{F,U}(y) - p_{G,U}(y) =& \mathbb{E}_U(\langle \nabla Q_d(F-y) - \nabla Q_d(G-y), H_U(G,1)\rangle) + \\ & + \mathbb{E}_U(\langle \nabla Q_d(F-y), H_U(F,1) - H_U(G,1)\rangle) \\ =& : I + J. \end{aligned}$$

Using (2.16) we obtain

$$|J| \leq C\,\gamma_{F,G,U}(p)^q n_{F,G,U}(p)^q m_U(p)^q \|\Delta_2(F,G)\|_{p,U}.$$

We study now the quantity I. For $\lambda \in [0,1]$ we denote $F_\lambda = G + \lambda(F-G)$ and we use Taylor's expansion to obtain

$$I = \sum_{k,j=1}^d R_{k,j} \quad \text{with} \quad R_{k,j} = \int_0^1 \mathbb{E}_U(\partial_k \partial_j Q_d(F_\lambda - y) H_{j,U}(G,1)(F-G)_k) d\lambda.$$

Let $V_{k,j} = H_{j,U}(G,1)(F-G)_k$. Using again the integration by parts formula (in respect to F_λ) we obtain

$$R_{k,j} = \int_0^1 \mathbb{E}_U\big(\partial_j Q_d(F_\lambda - y) H_{k,U}(F_\lambda, V_{k,j})\big) d\lambda.$$

Now, one has $\mathbb{E}_U((\det\sigma_{F_\lambda})^{-p}) \leq \gamma_{F,G,U}(p) < \infty$ for every $\lambda \in [0,1]$ and $p \geq 1$. So, we can use (2.18) and (2.17) with $F = F_\lambda$, and we get

$$\begin{aligned} |R_{k,j}| &\leq C\gamma_{U,F,G}(p)^q n_{U,F,G}(p)^q m_U(p)^q \,\|V_{k,j}\|_{1,p,U} \\ &\leq C'\gamma_{U,F,G}(p)^{q'} n_{U,F,G}(p')^{q'} m_U(p')^{q'} \,\|\Delta_2(F,G)\|_{p',U}. \end{aligned}$$

□

Example 2.3. *We give here an example of localizing function giving rise to a localizing random variable $\bar U$ that satisfies (2.1). For $a > 0$, set $\psi_a : \mathbb{R} \to \mathbb{R}_+$ as*

$$\psi_a(x) = 1_{|x|\leq a} + \exp\Big(1 - \frac{a^2}{a^2 - (x-a)^2}\Big) 1_{a<|x|<2a}. \tag{2.19}$$

Then $\psi_a \in C_b^1(\mathbb{R})$, $0 \leq \psi_a \leq 1$ and for every $p \geq 1$ one has

$$\sup_x |(\ln\psi_a(x))'|^p \psi_a(x) \leq \frac{4^p}{a_i^p} \sup_{t\geq 0}(t^{2p}e^{1-t}) < \infty.$$

For $\Theta_i \in \mathbb{D}^{1,\infty}$ and $a_i > 0$, $i = 1, ..., \ell$, we define

$$\bar U = \prod_{i=1}^{\ell} \psi_{a_i}(\Theta_i). \tag{2.20}$$

Then $\bar U \in \mathbb{D}^{1,\infty}$, $\bar U \in [0,1]$ and (2.1) holds. In fact, one has

$$\begin{aligned} |D\ln\bar U|^p\bar U =& \Big|\sum_{i=1}^{\ell}(\ln\psi_{a_i})'(\Theta_i)D\Theta_i\Big|^p \prod_{j=1}^{\ell}\psi_{a_j}(\Theta_j) \\ \leq& \Big(\sum_{i=1}^{\ell}|(\ln\psi_{a_i})'(\Theta_i)|^2\Big)^{p/2}\Big(\sum_{i=1}^{\ell}|D\Theta_i|^2\Big)^{p/2}\prod_{j=1}^{\ell}\psi_{a_j}(\Theta_j) \\ \leq& c_p\sum_{i=1}^{\ell}|(\ln\psi_{a_i})'(\Theta_i)|^p\psi_{a_i}(\Theta_i) \times |D\Theta|^p \\ \leq& C_p\sum_{i=1}^{\ell}\frac{1}{a_i^p}\,|D\Theta|^p \end{aligned}$$

for a suitable $C_p > 0$, so that

$$\mathbb{E}(|D\ln\bar U|^p\bar U) \leq C_p\sum_{i=1}^{\ell}\frac{1}{a_i^p} \times \mathbb{E}(|D\Theta|^p) \leq C_p\sum_{i=1}^{\ell}\frac{1}{a_i^p} \times \|\Theta\|_{1,p}^p < \infty. \tag{2.21}$$

Using the localizing function in (2.19) and by applying Proposition 2.2 we get the following result.

th-dist **Theorem 2.4.** *Assume that (2.1) holds. Let $F=(F_1,...,F_d)$ and $G=(G_1,...,G_d)$ with $F_i,G_i\in\mathbb{D}^{2,\infty}$ and such that for every $p\in\mathbb{N}$ one has*

$$\gamma_{F,U}(p):=1+\mathbb{E}_U((\det\sigma_F)^{-p}))<\infty \quad \textit{and} \quad \gamma_{G,U}(p):=1+\mathbb{E}_U((\det\sigma_G)^{-p}))<\infty.$$

Then under $\mathbb{P}_U$, the laws of F and G are absolutely continuous with respect to the Lebesgue measure, with continuous densities $p_{F,U}$ and $p_{G,U}$ respectively, and there exist a constant $C>0$ and two integers $p,q>1$ depending on d only such that

$$|p_{F,U}(y)-p_{G,U}(y)|\leq C\,(\gamma_{G,U}(p)\vee\gamma_{F,U}(p))^q n_{F,G,U}(p)^q m_U(p)^q\times\|\Delta_2(F,G)\|_{p,U} \tag{2.22}$$
Mall12

with $n_{F,G,U}(p)$ and $\Delta_2(F,G)$ given in (2.12) and $m_U(p)$ given in (2.1).

Proof. Set $R=F-G$. It is easy to check that for every $\lambda\in[0,1]$ one has

$$\det\sigma_{G+\lambda R}\geq\det\sigma_G-\alpha_d\,|DR|\,|DG|\,(1+|DF|+|DG|)^{d-1}, \tag{2.23}$$
alpha-d

for a suitable $\alpha_d>0$ depending on d only. For ψ_a as in (2.19), we define

$$V=\psi_{1/4}(H)\quad\text{with}\quad H=\frac{\alpha_d\,|DR|\,|DG|\,(1+|DF|+|DG|)^{d-1}}{\det\sigma_G}$$

so that if $V\neq0$ then $\det\sigma_{G+\lambda R}\geq\frac{1}{2}\det\sigma_G$. It follows that $\gamma_{F,G,UV}(p)\leq C\gamma_{G,U}(p)$, C denoting a suitable positive constant (which will vary in the following lines). We also have $m_{UV}(p)\leq C(m_U(p)+\mathbb{E}(UV|D\ln V|^p))$ and by (2.21) we have

$$\mathbb{E}(UV|D\ln V|^p)\leq C\,\|DH\|_{p,U}^p\leq C\,n_{F,G,U}(\bar p)^{\bar q}\gamma_{G,U}(\bar p)^{\bar q}$$

for some $\bar p,\bar q$, so that $m_{UV}(p)\leq C\,m_U(p)n_{F,G,U}(\bar p)^{\bar q}\gamma_{G,U}(\bar p)^{\bar q}$. So, we can apply (2.11) with localization UV and we get

$$|p_{F,UV}(y)-p_{G,UV}(y)|\leq C\big(\gamma_{F,U}(p)\vee\gamma_{G,U}(p)\big)^q n_{F,G,U}(p)^q m_U(p)^q\|\Delta_2(F,G)\|_{p,U}$$

We write now

$$|p_{F,U}(y)-p_{G,U}(y)|\leq|p_{F,UV}(y)-p_{G,UV}(y)|+\left|p_{F,U(1-V)}(y)\right|+\left|p_{G,U(1-V)}(y)\right|,$$

and we have already seen that the first addendum on the r.h.s. behaves as desired. So, it remains to see that also the remaining two terms have the right behavior. To this purpose, we use (2.9). So, we have

$$\left|p_{F,U(1-V)}(y)\right|\leq\gamma_{F,U}(p)^q n_{F,1,U}(p)^q m_U(p)^q\times\|1-V\|_{1,p,U}\,.$$

We recall that $1-V\neq0$ implies that $H\geq1/8$, so that

$$\begin{aligned}\|1-V\|_{1,p,U}^p&=\mathbb{E}_U(|1-V|^p)+\mathbb{E}_U(|DV|^p)\leq C\big(\mathbb{P}_U(H>1/8)+\mathbb{E}_U(V|D\ln V|^p)\big)\\&\leq C\big(\mathbb{E}_U(H^p)+\mathbb{E}_U(|DH|^p)\big)\end{aligned} \tag{2.24}$$
1-V

in which we have used (2.21). Now, one has

$$\mathbb{E}_U(|H|^p) \le C\,\gamma_{G,U}(\bar p)^{\bar q} n_{F,G,U}(\bar p)^{\bar q}\,\mathbb{E}_U(|D(F-G)|^{2p})^{1/2} \quad \text{and}$$
$$\mathbb{E}_U(|DH|^p) \le C\,\gamma_{G,U}(\bar p)^{\bar q} n_{F,G,U}(\bar p)^{\bar q}\,\big(\mathbb{E}_U(|D(F-G)|^{2p})^{1/2} + \mathbb{E}_U(|D^{(2)}(F-G)|^{2p})^{1/2}\big)$$

and by inserting above we get

$$\|1-V\|_{1,p,U} \le C\,\gamma_{G,U}(\bar p)^{\bar q} n_{F,G,U}(\bar p)^{\bar q}\,\|\Delta_2(F,G)\|_{2p,U}.$$

This gives

$$\big|p_{F,U(1-V)}(y)\big| \le C\big(\gamma_{F,U}(p)\vee\gamma_{G,U}(p)\big)^q n_{F,G,U}(p)^q m_U(p)^q \|\Delta_2(F,G)\|_{p,U}$$

for suitable constants $C>0$ and $p,q>1$. Similarly, we get

$$\big|p_{G,U(1-V)}(y)\big| \le C\big(\gamma_{F,U}(p)\vee\gamma_{G,U}(p)\big)^q n_{F,G,U}(p)^q m_U(p)^q \|\Delta_2(F,G)\|_{p,U}.$$

The statement now follows. □

3 Small perturbations of a Gaussian random variable

3.1 Preliminary estimates

We consider here a r.v. of the type $F = x + G + R \in \mathbb{R}^d$ where $R \in \mathbb{D}^{2,\infty}$ and

$$G = \sum_{j=1}^\infty \int_0^\infty h_j(s)dW_s^j$$

with $h_j : [0,+\infty) \to \mathbb{R}^d$ deterministic and square integrable. Then G is a centered Gaussian random variable of covariance matrix $M_G = (M_G^{k,p})_{k,p=1,\dots,d}$, with

$$M_G^{k,p} = \int_0^\infty \langle h^k(s), h^p(s)\rangle ds = \sum_{j=1}^\infty \int_0^\infty h_j^k(s)h_j^p(s)ds, \quad k,p = 1,\dots,d.$$

We assume that M_G is invertible and we denote by g_{M_G} the density of G that is

$$g_{M_G}(y) = \frac{1}{(2\pi)^{d/2}\sqrt{\det M_G}}\exp(-\langle M_G^{-1}y,y\rangle).$$

Our aim is to give estimates of the density of F in terms of g_{M_G}. To this purpose, we use a localizing r.v. U of the form (2.20).

exp-est **Proposition 3.1.** *Let ψ_a be the function in (2.19) and set*

$$U = \psi_{c_*^2/2}(|D\overline{R}|^2) \quad \text{with} \quad \overline{R} = M_G^{-1/2} R. \tag{3.1}$$

Law5'

where c_* is such that $\alpha_d(1+2d)^d c_*(1+c_*)^{d-1} \le 1/2$, α_d denoting the constant in (2.23). Then the following statements hold.

i) Under $\mathbb{P}_U$, the law of F has a smooth density $p_{F,U}$ and one has

$$\sup_{y\in\mathbb{R}^d} |p_{F,U}(y) - g_{M_G}(y-x)| \le \varepsilon(M_G, R), \quad y \in \mathbb{R}^d,$$

where

$$\varepsilon(M_G,R) := \frac{c_d}{\sqrt{\det M_G}}(1+\|\overline{R}\|_{2,q_d})^{\ell_d}\,\|\overline{R}\|_{2,q_d}\,.$$

Here c_d, q_d, ℓ_d are some universal positive constants depending on d only.

ii) If the law of F under $\mathbb{P}$ has a density p_F, then one has

$$p_F(y) \ge g_{M_G}(y-x) - \varepsilon(M_G,R), \quad y \in \mathbb{R}^d.$$

Proof. *i*) Suppose first that $x=0$ and $M_G = I$, I denoting the identity matrix, so that $\overline{R} = R$. We notice that $\det \sigma_G = 1$, which gives $\gamma_{G,U}(p) \le 2$ for every p, and $|DG|^2 = d$. Moreover, on the set $\{U \neq 0\}$ one has $|DR| \le c_*$ and by using (2.23) straightforward computations give

$$\det\sigma_F \ge 1 - \alpha_d(1+2d)^d|DR|(1+|DR|)^{d-1} \ge 1-\alpha_d(1+2d)^d c_*(1+c_*)^{d-1} \ge \frac{1}{2}.$$

It then follows that $\gamma_{F,U}(p) \le 1 + 2^p < \infty$ for every p. Moreover, by (2.21) one has $m_U(p) \le 1 + \|R\|_{2,2p}^p$. We can then apply Theorem 2.4 to the pair F and G, with localizing r.v. U. By straightforward computations and the use of the Meyer inequality, one has $n_{F,G,U}(p) \le C_p(1+\|R\|_{2,2p})^{2p}$ and $\|\Delta_2(F,G))\|_p \le C_p\|R\|_{2,2p}$, with $n_{F,G,U}(p)$ and $\Delta_2(F,G)$ given in (2.12). Therefore, (2.22) gives

$$|p_{F,U}(y) - p_{G,U}(y)| \le \bar{c}_1\,(1+\|R\|_{2,\bar{q}_1})^{\bar{\ell}_1}\,\|R\|_{2,\bar{q}_1} \quad \text{for every } y \in \mathbb{R}^d,$$

where $\bar{c}_1 > 0$ and $\bar{q}_1, \bar{\ell}_1 > 1$ are constants depending on d only. It remains to compare $p_{G,U}$ with $p_G = g_I$: from (2.9) (applied with $U=1$, $V = 1-U$ and $F=G$) we immediately have

$$|p_{G,U}(x) - p_G(x)| = p_{G,1-U}(x) \le C\gamma_{G,1}(p)^q n_{G,1}(p)^q m_1(p)^q\,\|1-U\|_{1,p} < C\|1-U\|_{1,p},$$

for a suitable $C>0$ and $p>1$ depending on d only. Now, recalling that $U \neq 1$ for $|DR| > c_*/\sqrt{2}$, as already seen in the proof of Theorem 3.3 (see (2.24)) we have

$$\|1-U\|_{1,p}^p \le C\big(\mathbb{E}(|DR|^{2p}) + \mathbb{E}(|D|DR|^2|^p)\big),$$

so that

$$|p_{G,U}(y) - p_G(y)| \le \bar{c}_2(1+\|R\|_{2,\bar{q}_2})^{\bar{\ell}_2}\,\|R\|_{2,\bar{q}_1} \quad \text{for every } y \in \mathbb{R}^d,$$

and the statement follows. As for the general case, it suffices to apply the already proved estimate to $\overline{F} = M_G^{-1/2}(F-x)$, $\overline{G} = M_G^{-1/2}G$ and $\overline{R} = M_G^{-1/2}R$ and then to use the change of variable theorem.

ii) It immediately follows from $p_F(y) \ge p_{F,U}(y)$. $\square$

3.2 Main results

positivity

In this section, we consider a time interval of the type $[T-\delta, T]$, where $T>0$ is a fixed horizon and $0<\delta\le T$, and we use the Malliavin calculus with respect to $W_s, s\in[T-\delta,T]$. In particular, we take conditional expectations with respect to $\mathcal{F}_{T-\delta}$. Therefore, for $V=(V^1,...,V^d)$, $V_i\in\mathbb{D}^{L,p}$, we define the following conditional Malliavin Sobolev norms:

$$\|V\|^p_{\delta,L,p}=\mathbb{E}(|V|^p\mid\mathcal{F}_{T-\delta})+\sum_{l=1}^{L}\mathbb{E}\big(|D^{(l)}V|^p\,\big|\,\mathcal{F}_{T-\delta}\big). \tag{3.2}$$

cond-mall-norm

Let F denote a d-dimensional functional on the Wiener space which is measurable w.r.t. $\mathcal{F}_T$ and assume that for $\delta\in(0,T]$ the following decomposition holds:

$$F=F_{T-\delta}+G_\delta+R_\delta \tag{3.3}$$

deco

where $F_{T-\delta}$ is measurable w.r.t. $\mathcal{F}_{T-\delta}$, $R_\delta\in(\mathbb{D}^{2,\infty})^d$ and

$$C_\delta=\sum_{k=1}^{\infty}\int_{T-\delta}^{T}h_\delta^k(s)dW_s^k.$$

Here $h^k_\delta(s), s\in[T-\delta,T]$ are progressively measurable processes such that $h^k_\delta(s)$ is $\mathcal{F}_{T-\delta}$-measurable for every $s\in[T-\delta,T]$ and $\sum_{k=1}^\infty\int_{T-\delta}^T|h^k_\delta(s)|^2ds<\infty$ a.s. In particular, conditionally on $\mathcal{F}_{T-\delta}$, the random variable G_δ is centered and Gaussian with covariance matrix

$$C^{ij}_\delta=\sum_{k=1}^{\infty}\int_{T-\delta}^{T}h^{k,i}_\delta(s)h^{k,j}_\delta(s)ds\quad 1\le i,j\le d.$$

On the set $\{\det C_\delta\neq 0\}\in\mathcal{F}_{T-\delta}$, we define the (random) norm

$$|x|_\delta:=|C_\delta^{-1/2}x|,\quad x\in\mathbb{R}^d$$

and for $q\in\mathbb{N}$, we consider the following (random) quantity

$$\theta_{\delta,q}=\|C_\delta^{-1/2}R_\delta\|_{\delta,2,q}. \tag{3.4}$$

theta

Set now $\overline{\mathbb{P}}_\delta(\omega,\cdot)$ the measure induced by $\overline{\mathbb{E}}_\delta(\omega,X)=\mathbb{E}(X\psi(|DR_\delta|^2)\,|\,\mathcal{F}_{T-\delta})(\omega)$, where $\psi=\psi_{1/8}$ is as in (3.1). By developing in a conditional form the arguments as in the proof of Proposition 3.1, on the set $\{\det C_\delta\neq 0\}$ one gets that under $\overline{\mathbb{P}}_\delta(\omega,\cdot)$ the law of F has a regular density w.r.t. the Lebesgue measure. Therefore, there exists a function $\bar p_{F,\delta}(\omega,z)$ which is regular as a function of z and such that

$$\mathbb{E}\big(f(F)\psi(|DR_\delta|^2)\,|\,\mathcal{F}_{T-\delta}\big)(\omega)=\int f(z)\bar p_{F,\delta}(\omega,z)dz,\quad \omega\in\{\det C_\delta\neq 0\} \tag{3.5}$$

pbardelta0

for any measurable and bounded function f.

Now, let us introduce the following sets: for $y \in \mathbb{R}^d$ and $r > 0$, we define

$$\Gamma_{\delta,r}(y) = \big\{|F_{T-\delta} - y|_\delta \le r\big\} \cap \big\{\det C_\delta \neq 0\big\} \cap \big\{\theta_{\delta,q_d} \le a_d\, e^{-r^2}\big\} \tag{3.6}$$ gamma0

$$\widetilde{\Gamma}_{\delta,r}(y) = \big\{|F_{T-\delta} - y|_\delta \le r/2\big\} \cap \big\{\det C_\delta \neq 0\big\} \cap \big\{\theta_{\delta,q_d} \le a_d\, e^{-r^2}\big\}, \tag{3.7}$$ gamma

where

$$a_d = \frac{1}{c_d 2^{\ell_d+1}(2\pi)^{d/2}}$$

and q_d, ℓ_d and c_d are the universal constants defined in (i) of Proposition 3.1. Then we have

lemma-pos **Lemma 3.2.** *For $\delta \in (0,T]$, let decomposition (3.3) hold and for $y \in \mathbb{R}^d$, $r > 0$, let $\Gamma_{\delta,r}(y)$ be the set in (3.6). Then for every non negative and measurable function $f : \mathbb{R}^d \to \mathbb{R}$ and $\omega \in \{\det C_\delta \neq 0\}$ one has*

$$\mathbb{E}(f(F) \,|\, \mathcal{F}_{T-\delta})(\omega) \ge \frac{e^{-r^2}}{2(2\pi)^{d/2}}\, (\det C_\delta)^{-1/2} \int f(z) \mathbf{1}_{\Gamma_{\delta,r}(z)} dz.$$

Proof. Let $\omega \in \{\det C_\delta \neq 0\}$. By using (3.5), for any measurable and non negative function f we have

$$\begin{aligned}\mathbb{E}(f(F)\mathcal{F}_{T-\delta})(\omega) &\ge \mathbb{E}(f(F)\psi(|DR_\delta|^2) \mid \mathcal{F}_{T-\delta})(\omega) = \int f(z)\overline{p}_{F,\delta}(\omega,z)dz \\ &\ge \int f(z)\overline{p}_{F,\delta}(\omega,z)\mathbf{1}_{\Gamma_{\delta,r}(z)}dz\end{aligned}$$

Using Proposition 3.1 in a conditional form (with respect to $\mathcal{F}_{T-\delta}$) we obtain

$$\overline{p}_{F,\delta}(\omega,z) \ge g_{C_\delta(\omega)}(z - F_{T-\delta}(\omega)) - \varepsilon(C_\delta(\omega), R_\delta)(\omega)$$

where, by using (3.4),

$$\begin{aligned}\varepsilon(C_\delta, R_\delta)(\omega) &\le \frac{c_d}{\sqrt{\det C_\delta}}(1 + \|C_\delta^{-1/2} R_\delta\|_{\delta,2,q_d})^{\ell_q} \|C_\delta^{-1/2} R_\delta\|_{\delta,2,q_d} \\ &= \frac{c_d}{\sqrt{\det C_\delta}}(1 + \theta_{\delta,q_d})^{\ell_q}\theta_{\delta,q_d}.\end{aligned}$$

If $\omega \in \Gamma_{\delta,r}(z)$ then $\theta_{\delta,q_d} \le a_d e^{-r^2} \le 1$ so that

$$\varepsilon(C_\delta, R_\delta)(\omega) \le \frac{1}{2} \times \frac{1}{(2\pi)^{d/2}\sqrt{\det C_\delta}}\, e^{-r^2}.$$

For $\omega \in \Gamma_{\delta,r}(z)$ we also have

$$\big\langle C_\delta^{-1}(F_{T-\delta} - z), F_{T-\delta} - z\big\rangle = |F_{T-\delta} - z|_\delta^2 \le r^2$$

so that

$$g_{C_\delta}(z - F_{T-\delta}) \ge \frac{1}{(2\pi)^{d/2}\sqrt{\det C_\delta}} e^{-r^2}.$$

Then, by the choice of $a_d e^{-r^2}$ we obtain

$$\overline{p}_{F,\delta}(\omega, z) \geq \frac{1}{(2\pi)^{d/2}\sqrt{\det C_\delta}} e^{-r^2} - \varepsilon(C_\delta, R_\delta)(\omega) \geq \frac{1}{2(2\pi)^{d/2}\sqrt{\det C_\delta}} e^{-r^2}.$$

We conclude that

$$\mathbb{E}(f(F) \mid \mathcal{F}_{T-\delta})(\omega) \geq \frac{1}{2e^{r^2}(2\pi)^{d/2}} \int f(z)(\det C_\delta)^{-1/2} 1_{\Gamma_{\delta,r}(z)} dz.$$

□

We are now ready for our main result. It involves the concept of "local densities", that we define as follows: we say that the law of a r.v. F taking values on $\mathbb{R}^d$ admits a local density around $y \in \mathbb{R}^d$ if there exists an open neighborhood V_y of y such that the restriction of the law of F in V_y is absolutely continuous w.r.t. the Lebesgue measure Leb_d on $\mathbb{R}^d$. So, we have:

th-pos **Theorem 3.3.** *For $\delta \in (0, T]$, let decomposition (3.3) hold and for $y \in \mathbb{R}^d$, $r > 0$, assume that*

$$\mathbb{P}(\widetilde{\Gamma}_{\delta,r}(y)) > 0,$$

where $\widetilde{\Gamma}_{\delta,r}(y)$ is the set in (3.7). Then there exists $\eta > 0$ and $c(y) > 0$ such that for every Borel measurable set $A \subset B_\eta(y)$ one has

$$\mathbb{P}(F \in A) \geq c(y)\mathrm{Leb}_d(A).$$

As a consequence, if the law of F admits a local density p_F around y then one has

$$p_F(x) \geq c(y) \quad \text{for a.e. } x \in B_\eta(y).$$

Proof. For $\varepsilon > 0$, set

$$\widetilde{\Gamma}_{\delta,r,\varepsilon}(y) = \big\{|F_{T-\delta} - y|_\delta \leq r/2\big\} \cap \big\{\det C_\delta \geq \varepsilon\big\} \cap \big\{\theta_{\delta,q_d} \leq a_d\, e^{-r^2}\big\}.$$

If $\mathbb{P}(\widetilde{\Gamma}_{\delta,r}(y)) > 0$ then there exists $\varepsilon > 0$ such that $\mathbb{P}(\widetilde{\Gamma}_{\delta,r,\varepsilon}(y)) > 0$. On the set $\{\det C_\delta \geq \varepsilon\}$, one has

$$|\xi|_\delta \leq \varepsilon^{-d/2}|\xi|, \quad \xi \in \mathbb{R}^d.$$

Taking $\eta = \varepsilon^{d/2} r/2$, one immediately has

$$\widetilde{\Gamma}_{\delta,r,\varepsilon}(y) \subset \Gamma_{\delta,r}(x) \quad \text{for every } x \in B_\eta(y)$$

where $\Gamma_{\delta,r}(x)$ is the set in (3.6). Therefore, by applying Lemma 3.2, for every measurable and bounded function f whose support is included in $B_\eta(y)$ one has

$$\mathbb{E}(f(F) \,|\, \mathcal{F}_{T-\delta})(\omega) \geq \frac{1}{2e^{r^2}(2\pi)^{d/2}}\, (\det C_\delta)^{-1/2}\, \mathbf{1}_{\Gamma_{\delta,r,\varepsilon}(y)} \int_{B_\eta(y)} f(x)dx,$$

and by passing to the expectation one gets the result with

$$c(y) = \frac{1}{2e^{r^2}(2\pi)^{d/2}}\, \mathbb{E}((\det C_\delta)^{-1/2}\mathbf{1}_{\Gamma_{\delta,r,\varepsilon}(y)}) > 0.$$

□

4 Examples

We apply now Theorem 3.3 to two cases in which a support theorem is available and we give results for the strict positivity and lower bounds for the density which involve suitable local or global non degeneracy conditions on the skeleton.

4.1 Ito processes

We consider here a process $Z_t = (X_t, Y_t)^*$, taking values on $\mathbb{R}^d \times \mathbb{R}^n$, which solves the following stochastic differential equation: as $t \leq T$,

$$\begin{aligned} X_t = &\ x_0 + \sum_{j=1}^m \int_0^t \sigma_j(X_t, Y_t) dW_t^j + \int_0^t b(X_t, Y_t) dt \\ Y_t = &\ y_0 + \sum_{j=1}^m \int_0^t \alpha_j(X_t, Y_t) dW_t^j + \int_0^t \beta(X_t, Y_t) dt. \end{aligned} \tag{4.1}$$

We are interested in dealing with strict positivity and/or lower bounds for the probability density function of one component at a fixed time, say X_T, as a consequence of Theorem 3.3. This is a case in which a support theorem is available, and we are going to strongly use it. For diffusion processes (that is, if we deal with Z_T and not with X_T only), we get an example which is essentially the same as in the paper of Ben Arous and Leandre [8] and in the paper of Aida, Kusouka and Stroock [1]. Concerning the lower bounds, we will use lower estimates for the probability that Ito processes remains in a tube around a path proved in Bally, Fernández and Meda in [6].

So, in (4.1) we assume that $\sigma_j, b \in C_b^4(\mathbb{R}^{d+n}; \mathbb{R}^d)$ and $\alpha_j, \beta \in C_b^4(\mathbb{R}^{d+n}; \mathbb{R}^n)$, $j = 1, \dots, m$, which implies that $X_t^\ell, Y_t^i \in \mathbb{D}^{2,\infty}$ for all ℓ and i.

For $\phi \in L^2([0,T]; \mathbb{R}^m)$, let $z_t(\phi) = (x_t(\phi), y_t(\phi))^*$ denote the skeleton associated to (4.1), i.e.

$$\begin{aligned} x_t(\phi) = &\ x_0 + \sum_{j=1}^m \int_0^t \sigma_j\big(x_t(\phi), y_t(\phi)\big) \phi_t^j \, dt + \int_0^t \overline{b}\big(x_t(\phi), y_t(\phi)\big) dt \\ y_t(\phi) = &\ y_0 + \sum_{j=1}^m \int_0^t \alpha_j\big(x_t(\phi), y_t(\phi)\big) \phi_t^j \, dt + \int_0^t \overline{\beta}\big(x_t(\phi), y_t(\phi)\big) dt, \end{aligned} \tag{4.2}$$

in which $\overline{b} = b - \frac{1}{2}\sum_{j=1}^m \partial_{\sigma_j}\sigma_j$ and $\overline{\beta} = \beta - \frac{1}{2}\sum_{j=1}^m \partial_{\alpha_j}\alpha_j$, where we have used the notation $(\partial_g f)^i = \langle \nabla f^i, g \rangle$.

For a fixed $x \in \mathbb{R}^d$, we set

$$\mathcal{C}(x) = \{\phi \in L^2([0,T]; \mathbb{R}^m) \,:\, x_T(\phi) = x\}. \tag{4.3}$$

We finally consider the following set of functions: for fixed $\mu \geq 1$ and $h > 0$,

$$L(\mu, h) = \{f \,:\, [0,T] \to \mathbb{R}_+ \,;\, f_t \leq \mu f_s \text{ for all } t, s \text{ such that } |t-s| \leq h\}. \tag{4.4}$$

We have

Theorem 4.1. *Let $Z=(X,Y)^*$ denote the solution of (4.1), with $\sigma_j, b\in C_b^4(\mathbb{R}^{d+n};\mathbb{R}^d)$ and $\alpha_j,\beta\in C_b^4(\mathbb{R}^{d+n};\mathbb{R}^n)$, $j=1,\dots,m$. Let $x\in\mathbb{R}^d$ be fixed and suppose that $\mathcal{C}(x)\neq\emptyset$. For $\phi\in\mathcal{C}(x)$, let $z_t(\phi)=(x_t(\phi),y_t(\phi))^*$ be as in (4.2).*

i) Suppose there exists $\phi\in\mathcal{C}(x)$ such that $\sigma\sigma^*(x,y_T(\phi))>0$. Then there exists $\eta>0$ and $c(x)>0$ such that for every Borel measurable set $A\subset B_\eta(x)$ one has

$$\mathbb{P}(X_T\in A)\geq c(x)\mathrm{Leb}_d(A).$$

In particular, if X_T admits a local density p_{X_T} around x then $p_{X_T}\geq c(x)>0$ a.e. on the ball $B_\eta(x)$.

ii) Suppose there exists $\phi\in\mathcal{C}(x)$ such that $|\partial x_t(\phi)|\in L(\mu,h)$, for some $\mu\geq 1$ and $h>0$, and

$$\sigma\sigma^*\big(x_t(\phi),y\big)\geq\lambda_*>0\quad \textit{for all } t\in[0,T] \textit{ and } y\in\mathbb{R}^n.$$

Then if the law of X_T admits a continuous local density p_{X_T} around x one has

$$p_{X_T}(x)\geq\Upsilon\exp\Big[-Q\Big(\Psi+\frac{1}{\lambda_*}\int_0^T|\partial_t x_t(\phi)|dt\Big)\Big],$$

where Υ, Q, Ψ are all positive constants depending on d,T,μ,h,λ_* and vector fields σ_j,α_j, $j=1,\dots,m$, and b,β.

In next Proposition 4.3 we study the existence of a local density and we prove in particular that under the requirement in part *ii*), the local density really exists. Actually, a little bit more work would show that the non degeneracy condition (2.3) holds and by Lemma 2.1 the local density is indeed continuous. But we are not interested here to enter in these technical arguments.

Proof of Theorem 4.1. *i*) We take $0<\delta\leq T$ and we consider the decomposition $X_T=X_{T-\delta}+G_\delta+R_\delta$, where

$$\begin{aligned}G_\delta&=\sum_{j=1}^m\int_{T-\delta}^T\sigma_j(X_{T-\delta},Y_{T-\delta})dW_t^j\\ R_\delta&=\sum_{j=1}^m\int_{T-\delta}^T\big(\sigma_j(X_t,Y_t)-\sigma_j(X_{T-\delta},Y_{T-\delta})\big)dW_t^j+\int_{T-\delta}^T b(X_t,Y_t)dt.\end{aligned}$$

Conditionally on $\mathcal{F}_{T-\delta}$, the covariance matrix of the Gaussian r.v. G_δ is

$$C_\delta=\sigma\sigma^*(X_{T-\delta},Y_{T-\delta})\delta.$$

So, we are in the framework studied in Section 3 and we proceed in order to apply Theorem 3.3: *i*) is proved as soon as we find $\delta,r>0$ such that $\mathbb{P}(\widetilde{\Gamma}_{\delta,r}(x))>0$.

For $\phi\in\mathcal{C}(x)$, we denote $z^\phi(x)=(x,y_T(\phi))$ and we take ϕ such that $\sigma\sigma^*(z^\phi(x))>0$. We denote by $\lambda_*>0$ the lower eigenvalue of $\sigma\sigma^*(z^\phi(x))$. Then, there exists $\varepsilon>0$ such that

$$\sigma\sigma^*(z)\geq\frac{\lambda_*}{2}I_d\quad\text{for every } z \text{ such that } |z-z^\phi(x)|<\varepsilon.$$

For a fixed $\delta \in (0,T]$, we have $|z^\phi(x) - z_{T-\delta}(\phi)| = |z_T(\phi) - z_{T-\delta}(\phi)| \le C(1 + \|\phi\|_2)\sqrt{\delta} = C_\phi\sqrt{\delta}$, so that if $|Z_{T-\delta} - z_{T-\delta}(\phi)| < C_\phi\sqrt{\delta}$ then $|Z_{T-\delta} - z^\phi(x)| < 2C_\phi\sqrt{\delta}$. We choose δ_0 such that $2C_\phi\sqrt{\delta} < \varepsilon$ for all $\delta < \delta_0$. So, if $|Z_{T-\delta} - z_{T-\delta}(\phi)| < C_\phi\sqrt{\delta}$ we get

$$C_\delta \ge \frac{\lambda_*}{2}\,\delta I_d$$

and in particular,

$$\begin{aligned}|X_{T-\delta} - x|_\delta = |C_\delta^{-1/2}(X_{T-\delta} - x)| &\le \Big(\frac{2}{\lambda_*\delta}\Big)^{1/2} |X_{T-\delta} - x| \\ &\le \Big(\frac{2}{\lambda_*\delta}\Big)^{1/2} |Z_{T-\delta} - z^\phi(x)| < \frac{2\sqrt{2}\,C_\phi}{\sqrt{\lambda_*}} =: \frac{r}{2}.\end{aligned}$$

Moreover, for $q \ge 2$, a standard reasoning gives

$$\begin{aligned}\|R_\delta\|_{\delta,2,q}^q &= \mathbb{E}(|R_\delta|^q \mid \mathcal{F}_{T-\delta}) + \mathbb{E}\Big(\sum_{l=1}^{2}\Big(\int_{[T-\delta,T]^l} \big|D^{(l)}_{s_1\ldots s_l} R_\delta\big|^2\, ds_1\ldots ds_l\Big)^{q/2} \mid \mathcal{F}_{T-\delta}\Big) \\ &\le (C_{1,q}\delta)^q,\end{aligned}$$

so that

$$\theta_{\delta,q} = \|C_\delta^{-1/2} R_\delta\|_{\delta,2,q} \le \frac{1}{\sqrt{\lambda_*\delta}}\,\|R_\delta\|_{\delta,2,q} \le C_{2,q}\sqrt{\delta}. \tag{4.5}$$

theta-th-ito

We take $\delta < \delta_0$ in order that $C_{2,q}\sqrt{\delta} < a_d e^{-r^2}$. For such a δ we get that $\{|Z_{T-\delta} - z_{T-\delta}(\phi)| < C_\phi\sqrt{\delta}\} \subset \widetilde{\Gamma}_{\delta,r}(x)$ and by the support theorem one has $\mathbb{P}(|Z_{T-\delta} - z_{T-\delta}(\phi)| < C_\phi\sqrt{\delta}) > 0$, so that $\mathbb{P}(\widetilde{\Gamma}_{\delta,r}(x)) > 0$.

ii) For $\xi : [0,T] \to \mathbb{R}^d$ and $R > 0$, we set

$$\tau_R^\phi(\xi) = \inf\{t : |\xi_t - x_t(\phi)| \ge R\}.$$

We know that there exists $\phi \in \mathcal{C}(x)$ and $\varepsilon > 0$ such that if $\tau_\varepsilon^\phi(\xi) > T$ then

$$\sigma\sigma^*(\xi_t, y) \ge \lambda_* I_d$$

for any $t \in [0,T]$ and $y \in \mathbb{R}^n$. So, on the set $\{\tau_\varepsilon^\phi(X) > T\}$ one gets $C_\delta \ge \lambda_*\delta I_d$. Moreover, if $\tau_\varepsilon^\phi(X) > T$ then for $0 < \delta < T$

$$\begin{aligned}|X_{T-\delta} - x| = |X_{T-\delta} - x_T(\phi)| &\le |X_{T-\delta} - x_{T-\delta}(\phi)| + |x_{T-\delta}(\phi) - x_T(\phi)| \\ &< \varepsilon + \int_{T-\delta}^T |\partial_t x_t(\phi)| dt.\end{aligned}$$

Since again (4.5) holds, we take $\delta < T$ such that $\int_{T-\delta}^T |\partial_t x_t(\phi)| dt < \varepsilon$ and $\theta_{\delta,q_d} \le a_d e^{-(2\varepsilon)^2}$. Therefore, $\{\tau_\varepsilon^\phi(X) > T\} \subset \Gamma_{\delta,2\varepsilon}(x)$ and by using Lemma 3.2 we get

$$p_{X_T}(x) \ge \frac{1}{2(2\pi\lambda_*\delta)^{d/2} e^{4\varepsilon^2}}\,\mathbb{P}\big(\tau_\varepsilon^\phi(X) > T\big) \equiv \Upsilon \times \mathbb{P}\big(\tau_\varepsilon^\phi(X) > T\big).$$

Now, the hypothesis allow one to use Theorem 1 in Bally, Fernández and Meda [6]: one has

$$\mathbb{P}(\tau_\varepsilon^\phi(X) > T) \geq \exp\Big(-Q\Big(\Psi + \frac{1}{\lambda_*}\int_0^T |\partial_t x_t(\phi)|dt\Big)\Big)$$

and the statement holds. $\square$

grushin **Example 4.2.** *Let $n \geq 1$ and $k \geq 0$ be fixed integers and let (X,Y) be the 2-dimensional process solution to*

$$\begin{aligned} X_t &= x_0 + \int_0^t Y_s^n \, dW_s^1 + \int_0^t Y_s^k \, ds, \\ Y_t &= y_0 + W_t^2, \end{aligned}$$

W denoting a Brownian motion on $\mathbb{R}^2$. The pair (X,Y) then follows the well-known Grushin diffusion. Here, we are interested in the study of the component X only, because this gives an example in between the two cases studied in Theorem 4.1. In fact, one has $\sigma\sigma^*(x,y) = y^{2n}$, and this vanishes as $y=0$, so there is no hope that part ii) holds. Nevertheless, i) is always true. In fact, since the strong Hörmander condition holds for the diffusion pair (X,Y), the law of (X_T, Y_T) has a smooth density on $\mathbb{R}^2$, so that X_T has a smooth density as well. Moreover, the associated skeleton is given by

$$\begin{aligned} x_t(\phi) = &\quad x_0 + \int_0^t y_t^n(\phi)\phi_t^1 \, dt + \frac{1}{2}\int_0^t \big(2y_t^k(\phi) - n\, y_t^{2n-1}(\phi)\big)dt \\ y_t(\phi) = &\quad y_0 + \int_0^t \phi_t^2 \, dt, \end{aligned}$$

so it is clear that for every $x \in \mathbb{R}$ one has $\mathcal{C}(x) \neq \emptyset$ and one can choose $\phi \in \mathcal{C}(x)$ such that $\sigma\sigma^*(x, y_T(\phi)) > 0$, that gives $p_{X_T}(x) > 0$.

We propose now a sufficient condition for the existence of a local density, that in particular says that under the hypothesis of *ii*) in Theorem 4.1, a local density really exists.

prop-ito **Proposition 4.3.** *Set*

$$\mathcal{O} = \{x \in \mathbb{R}^d \, : \, \mathbb{P}(\sigma\sigma^*(x, Y_T) > 0) = 1\}.$$

Then for every $x \in \mathcal{O}$ the law of X_T admits a local density p_{X_T} around x. As a consequence, if $x \in \mathcal{O}$ is such that $\mathcal{C}(x) \neq \emptyset$ and for some $\phi \in \mathcal{C}(x)$ one has $\sigma\sigma^*(x, y_T(\phi)) > 0$, then the local density p_{X_T} is a.e. strictly positive around x.

Proof. For $x \in \mathcal{O}$, set $D_x = \{y \in \mathbb{R}^n \, : \, \sigma\sigma^*(x,y) > 0\}$. D_x is an open set, so there exist a sequence $\{y_i\} \subset \mathbb{R}^n$ and a sequence $\{r_i\}_i \subset \mathbb{R}_+$ such that $D_x = \cup_{i\in\mathbb{N}} B_{\frac{1}{2}r_i}(y_i)$ and $B_{r_i}(y_i) \subset D_x$. Moreover $\sigma\sigma^*(\bar{x}, y) \geq \lambda_i > 0$ for every $y \in B_{r_i}(y_i)$ and $\bar{x} \in B_r(x)$. For any fixed i, we consider a localizing r.v. U_i of the form (2.20): we set

$$U_i = \psi_{r^2}(|X_T - x|^2)\psi_{r_i^2}(|Y_T - y_i|^2).$$

By (2.21), U_i is a good localizing r.v. (that is, (2.1) holds), and we set $d\mathbb{P}_i = U_i d\mathbb{P}$. We also notice that if $U_i \neq 0$ then $\sigma\sigma^*(X_T, Y_T) \geq \lambda_i > 0$. This property allows one to use a standard argument showing that the Malliavin covariance matrix of $F = X_T$ has finite inverse moments of any order with respect to $\mathbb{P}_i$, which means that (2.3) holds. So, we can use Lemma 2.1 and we can conclude that the law of X_T with respect to $\mathbb{P}_i$ is absolutely continuous with respect to the Lebesgue measure. Take now $A \subset B_{r/2}(x)$ a set of Lebesgue measure equal to zero. Since $x \in \mathcal{O}$ we have $\mathbb{P}(Y_T \in D_x) = 1$ so

$$\begin{aligned}\mathbb{P}(X_T \in A) = \mathbb{P}(X_T \in A, Y_T \in D_x) &\leq \sum_i \mathbb{P}(X_T \in A, Y_T \in B_{\frac{1}{2}r_i}(y_i)) \\ &= \sum_i \mathbb{P}_i(X_T \in A, Y_T \in B_{\frac{1}{2}r_i}(y_i))\end{aligned}$$

the last equality being true because $\psi_{r^2}(|X_T - x|^2)\psi_{r_i^2}(|Y_T - y_i|^2) = 1$ if $X_T \in A$ and $Y_T \in B_{\frac{1}{2}r_i}(y_i)$. Since the law of X_T under $\mathbb{P}_i$ is absolutely continuous with respect to the Lebesgue measure we obtain $\mathbb{P}_i(X_T \in A, Y_T \in B_{\frac{1}{2}r_i}(y_i)) = 0$ for every i, and this proves that a local density p_{X_T} around x exists. The final statement comes now immediately from Theorem 4.1. $\square$

Example 4.4. *Consider the diffusion process*

$$\begin{aligned}X_t^1 &= x_0^1 + \int_0^t \alpha(|X_s|)\,|Y_s| \circ dW_s^1 + \int_0^t |X_s| \circ dW_s^3, \\ X_t^2 &= x_0^2 + \int_0^t \alpha(|X_s|)\,|Y_s| \circ dW_s^2 + \int_0^t |X_s| \circ dW_s^3, \\ dY_t &= y_0 + \int_0^t \beta(X_s) \circ dW_t^4\end{aligned}$$

where W is a standard Brownian motion taking values on $\mathbb{R}^4$ and α, β are C_b^4 functions. We suppose that $\{r : \alpha(r) \neq 0\} = B_1(0)$ and that $\beta(x_0) \neq 0$, the latter requirement ensuring in particular that the law of Y_T has a density. Therefore, for every $x \in B_1(0)$ one has $\mathbb{P}(\sigma\sigma^*(x, Y_T) > 0) = 1$ and by applying Proposition 4.3 one gets that X_T has a local density around every point in $B_1(0)$. Now, in order to study its positivity property, let us write down the associated skeleton: for a square integrable control path ϕ, one has

$$\begin{aligned}x_t^1(\phi) =& x_0^1 + \int_0^t \alpha(|x_s(\phi)|)\,|y_t(\phi)|\,\phi_s^1 ds + \int_0^t |x_s(\phi)|\,\phi_s^3 ds \\ x_t^2(\phi) =& x_0^2 + \int_0^t \alpha(|x_s(\phi)|)\,|y_t(\phi)|\,\phi_s^2 ds + \int_0^t |x_s(\phi)|\,\phi_s^3 ds, \\ y_t(\phi) =& y_0 + \int_0^t \beta(x_s(\phi)))\phi_s^4 ds.\end{aligned}$$

We recall that the support theorem of Stroock and Varadhan asserts that the law of $(X_t^1, X_t^2, Y_t)_{t\geq 0}$ *is the closure (with respect to the uniform norm) of the points of the skeleton as above. Notice that if* $|x_t^1(\phi)| \geq 1$ *then* $\alpha(|x_t(\phi)|) = 0$ *and so* $\partial_t x_t^1(\phi) = |x_t(\phi)|\, \phi_t^3 = \partial_t x_t^2(\phi)$. *This means that outside the unit ball the skeleton* $(x_t^1(\phi), x_t^2(\phi))$ *may travel on a line which is parallel to the principal diagonal (i.e.* $x^1 = x^2$*), but only on this line. If* $|x_t^1(\phi)| < 1$ *then one may use the controls* ϕ^1 *and* ϕ^2 *and then* $(x_t^1(\phi), x_t^2(\phi))$ *may travel in any direction inside the open unit ball. Having this in mind, we define the strip* $S = \{(x^1, x^2) : |x^1 - x^2| < \sqrt{2}\}$ *and thanks to the above discussion and the support theorem we have the following three cases.*

1. $x_0 \notin S$. *Here, for every* s *the law of* X_s *is concentrated on the line which is parallel to the principal diagonal and contains* x_0. *In particular,* $\alpha(|X_s|) = 0$ *a.s. for every* s*, so* X *is actually a diffusion process satisfying*

$$X_t^1 = x_0^1 + \int_0^t |X_s| \circ dW_s^3, \quad X_t^2 = x_0^2 + \int_0^t |X_s| \circ dW_s^3.$$

2. $x_0 \in S$ but $x_0 \notin B_1(0)$. *Here the support of the law of* X_T *is the whole* S. *By using Proposition 4.3, we can say that* X_T *has a local density around any point in* $B_1(0)$ *and moreover, there exists a version of the local density which is strictly positive in the ball. But we have no information outside the ball.*

3. $x_0 \in B_1(0)$. *We can assert the same statements as in case 2 but with some refinements. In fact, here if* $y_0 \neq 0$ *then* $\alpha(x_0)y_0 \neq 0$*, so that the law of* X_T *has a smooth global density which is strictly positive on the unit ball* $B_1(0)$.

Concerning point ii) of Theorem 4.1, it does not apply except when $x_0, x \in B_1(0)$.

4.2 Diffusion processes satisfying a weak Hörmander condition: an example

In this section we treat an example of diffusion process which satisfies the weak Hörmander condition and has been recently studied in Bally and Kohatsu-Higa [5] (we are going to use the ideas and the estimates from that paper). Since lower bounds for the density have been already discussed in [5], we deal here only with the strict positivity. So, we give an application of our Theorem 3.3 in a case of degenerate diffusion coefficients.

We consider the diffusion process

$$X_t^1 = x^1 + \int_0^t \sigma_1(X_s)dW_s + \int_0^t b_1(X_s)ds, \quad X_t^2 = x^2 + \int_0^t b_2(X_s)ds \tag{4.6}$$

and we assume that $\sigma_1, b_1, b_2 \in C_b^\infty(\mathbb{R}^2; \mathbb{R})$. Actually, it suffices that they are four times differentiable - but we do not focus on this aspect here. Moreover, we fix some point $y \in \mathbb{R}^2$ and we assume that

$$|\sigma_1(y)| > c_* > 0 \quad \text{and} \quad |\partial_1 b_2(y)| > c_* > 0. \tag{4.7}$$

Let $\sigma = (\sigma_1, 0)^*$ and $b = (b_1, b_2)^*$. The Lie bracket $[\sigma, b]$ is computed as

$$[\sigma, b](x) = \partial_\sigma b(x) - \partial_b \sigma(x) = \begin{pmatrix} \sigma_1(x)\partial_1 b_1(x) - b_1(x)\partial_1\sigma_1(x) - b_2(x)\partial_2\sigma_1(x) \\ \sigma_1(x)\partial_1 b_2(x) \end{pmatrix}.$$

So assumption (4.7) is equivalent with the fact that $\sigma(y)$ and $[\sigma, b](y)$ span $\mathbb{R}^2$, and this is the weak Hörmander condition in y.
We set $\overline{b} = b - \frac{1}{2}\partial_\sigma \sigma$ and for a measurable function $\phi \in L^2([0,T], \mathbb{R})$ we consider the skeleton $x(\phi)$, i.e. the solution of the equation

$$x_t(\phi) = x + \int_0^t \Big(\sigma(x_s(\phi))\phi_s + \overline{b}(x_s(\phi))\Big)ds.$$

hor-prop **Proposition 4.5.** *Assume that $\sigma_1, b_1, b_2 \in C_b^\infty(\mathbb{R}^2)$ and (4.7) holds. Then the law of X_T has a local smooth density $p_T(x, \cdot)$ in a neighborhood of y. Moreover, if there exists a control $\phi \in L^2([0,T])$ such that $x_T(\phi) = y$ then $p_T(x,y) > 0$.*

Before starting with the proof of Proposition 4.5, let us consider the following decomposition: for $\delta \in (0, T]$, we set

$$F = X_T - x_T(\phi) \quad \text{and} \quad F = F_{T-\delta} + G_\delta + R_\delta \tag{4.8}$$ hor-dec

where $F_{T-\delta} = X_{T-\delta} - x_{T-\delta}(\phi)$ and

$$\begin{aligned}
G_\delta^1 &= \int_{T-\delta}^T \sigma_1(X_{T-\delta})dW_s, \quad G_\delta^2 = \int_{T-\delta}^T \partial_\sigma b_2(X_{T-\delta})(T-s)dW_s \\
R_\delta^1 &= \int_{T-\delta}^T \Big(\sigma_1(X_s) - \sigma_1(X_{T-\delta})\Big)dW_s + \int_{T-\delta}^T b_1(X_s)ds + \\
&\quad - \int_{T-\delta}^T \Big(\sigma_1\big(x_s(\phi)\big)\phi_s + \overline{b}_1\big(x_{T-\delta}(\phi)\big)\Big)ds \\
R_\delta^2 &= \int_{T-\delta}^T \Big(\partial_\sigma b_2(X_s) - \partial_\sigma b_2(X_{T-\delta})\Big)(T-s)dW_s + \delta\Big(b_2(X_{T-\delta}) - b_2\big(x_{T-\delta}(\phi)\big)\Big) + \\
&\quad + \int_{T-\delta}^T Lb_2(X_s)(T-s)ds - \int_{T-\delta}^T \Big(b_2\big(x_s(\phi)\big) - b_2\big(x_{T-\delta}(\phi)\big)\Big)ds
\end{aligned}$$

in which $L = \frac{1}{2}\sigma\sigma^*\partial_x^2 + b\partial_x$ denotes the infinitesimal generator of X.
The covariance matrix of the conditional (on $\mathcal{F}_{T-\delta}$) Gaussian r.v. G_δ is given by

$$C_\delta = \delta\sigma_1^2(X_{T-\delta}) \begin{pmatrix} 1 & \partial_1 b_2(X_{T-\delta})\frac{\delta}{2} \\ \partial_1 b_2(X_{T-\delta})\frac{\delta}{2} & (\partial_1 b_2)^2(X_{T-\delta})\frac{\delta^2}{3} \end{pmatrix}.$$

We need now some estimates which can be easily deduced from [5]. In order to be self contained, we propose here the following

hor-lemma **Lemma 4.6.** *Let $\rho_\delta^2 = \max(\delta, \int_{T-\delta}^T |\phi_s|^2 ds)$. Then, there exist $\delta_0 > 0$ such that for every $\delta < \delta_0$, on the set $\{|F_{T-\delta}| < \delta^{3/2}\rho_\delta\}$ the following properties hold:*

i) $\det C_\delta \geq c_1 \frac{\delta^4}{12}$;

ii) for every $\xi \in \mathbb{R}^2$, $|\xi|_\delta^2 \leq \frac{c_2}{\delta^3}\Big(\delta^2|\xi_1|^2 + |\xi_2|^2\Big)$; *in particular,* $|F_{T-\delta}|_\delta \leq c_2\rho_\delta$;

ii) for every $q \geq 2$, $\theta_{\delta,q} \leq L_q\rho_\delta$.

Here, c_1, c_2 *and* L_q *are suitable positive constants depending on* c_* *and upper bounds for* σ, b *and their derivatives up to order* 4, L_q *depending on* q *also, and we recall that* $|\xi|_\delta = |C_\delta^{-1/2}\xi|$.

Proof. First, by recalling that σ and b are bounded, for some positive constant C we have

$$|x_{T-\delta}(\phi) - x_T(\phi)| \leq C\rho_\delta^2$$

so that

$$|X_{T-\delta} - y| \leq |X_{T-\delta} - x_{T-\delta}(\phi)| + C\rho_\delta^2.$$

Therefore, we can choose δ_0 such that for all $\delta < \delta_0$ the following holds: if $|F_{T-\delta}| = |X_{T-\delta} - x_{T-\delta}(\phi)| < \delta^{3/2}\rho_\delta$ then

$$|\sigma_1(X_{T-\delta})| \geq c_* > 0 \quad \text{and} \quad |\partial_1 b_2(X_{T-\delta})| \geq c_* > 0.$$

Therefore,

$$\det C_\delta = \frac{(\sigma_1^2\partial_1 b_2)^2(X_{T-\delta})\delta^4}{12} \geq c_1\delta^4$$

and *i)* holds. Moreover, we have

$$C_\delta^{-1} = \frac{1}{(\sigma_1\partial_1 b_2)^2(X_{T-\delta})\delta^3}\begin{pmatrix} 4(\partial_1 b_2)^2(X_{T-\delta})\delta^2 & -6(\partial_1 b_2)(X_{T-\delta})\delta \\ -6(\partial_1 b_2)(X_{T-\delta})\delta & 12 \end{pmatrix}$$

so that for $\xi \in \mathbb{R}^2$,

$$\begin{aligned}|C_\delta^{-1/2}\xi|^2 = \langle C_\delta^{-1}\xi,\xi\rangle &= \frac{1}{(\sigma_1\partial_1 b_2)^2(X_{T-\delta})\delta^3}\Big(\big(2\partial_1 b_2(X_{T-\delta})\delta\xi_1 - 3\xi_2\big)^2 + 3\xi_2^2\Big)\\ &\leq \frac{C}{c_*^4\delta^3}\Big(\delta^2|\xi_1|^2 + |\xi_2|^2\Big)\end{aligned}$$

where C depends on σ and b. Then, if $|F_{T-\delta}| < \delta^{3/2}\rho_\delta$ one gets

$$|F_{T-\delta}|_\delta^2 = |C_\delta^{-1/2}F_{T-\delta}|^2 \leq \frac{C}{c_*^4\delta^3}\delta^3\rho_\delta^2(\delta^2+1) \leq c_2\rho_\delta^2$$

and *ii)* is proved. As for *iii)*, for $q \geq 2$ we have

$$\mathbb{E}\big(|C_\delta^{-1/2}R_\delta|^q\big) \leq \Lambda_q\Big(\mathbb{E}(|\delta^{-1/2}R_\delta^1|^q) + \mathbb{E}(|\delta^{-3/2}R_\delta^2|^q)\Big),$$

where Λ_q depends on q, c_*, σ and b. Now, by using the Burkholder inequality and the boundedness of the coefficients b and σ and of their derivatives, one has

$$\begin{aligned}\mathbb{E}(|\delta^{-1/2}R^1_\delta|^q) \le C_q\delta^{-q/2}\Big[&\mathbb{E}\Big(\Big|\int_{T-\delta}^T \big(\sigma_1(X_s)-\sigma_1(X_{T-\delta})\big)dW_s\Big|^q\Big)+\\ &+\mathbb{E}\Big(\Big|\int_{T-\delta}^T b_1(X_s)ds\Big|^q\Big)+\\ &+\mathbb{E}\Big(\Big|\int_{T-\delta}^T \Big(\sigma_1\big(x_s(\phi)\big)\phi_s+\bar b_1\big(x_{T-\delta}(\phi)\big)\Big)ds\Big|^q\Big)\Big]\\ \le C_qC\delta^{-q/2}\cdot\Big(\delta^q+\delta^{q/2}\Big(&\int_{T-\delta}^T|\phi_s|^2ds\Big)^{q/2}\Big)\le 2C_qC\rho_\delta^q\end{aligned}$$

where C_q depends on q only and C depends on the bounds of the diffusion coefficients. Similarly (in the following C denotes a suitable constant),

$$\begin{aligned}\mathbb{E}(|\delta^{-3/2}R^2_\delta|^q) \le C_q\delta^{-3q/2}\Big[&\mathbb{E}\Big(\Big|\int_{T-\delta}^T \big(\partial_\sigma b_2(X_s)-\partial_\sigma b_2(X_{T-\delta})\big)(T-s)dW_s\Big|^q\Big)+\\ &+\mathbb{E}\Big(\delta^q\big|b_2(X_{T-\delta})-b_2\big(x_{T-\delta}(\phi)\big)\big|^q\Big)+\\ &+\mathbb{E}\Big(\Big|\int_{T-\delta}^T Lb_2(X_s)(T-s)ds\Big|^q\Big)+\\ &+\Big|\int_{T-\delta}^T\Big(b_2\big(x_s(\phi)\big)-b_2\big(x_{T-\delta}(\phi)\big)\Big)ds\Big|^q\Big]\\ \le 2C_qC\delta^{-3q/2}&\Big(\delta^{2q}+\delta^q|F_{T-\delta}|^q+\delta^q\sup_{T-\delta\le s\le T}|x_s(\phi)-x_{T-\delta}(\phi)|^q\Big)\\ \le 2C_qC\delta^{-3q/2}&\Big(\delta^{2q}+\delta^q\cdot\delta^{3q/2}\rho_\delta^q+\delta^q\cdot\Big[\delta^q+\delta^{q/2}\cdot\Big(\int_{T-\delta}^T|\phi_s|^2ds\Big)^{q/2}\Big]\Big)\\ \le C_qC\rho_\delta^q.&\end{aligned}$$

The same arguments may be used to give upper estimates for the remaining terms in $\|C_\delta^{-1/2}R_\delta\|^q_{\delta,2,q}$ that contain the Malliavin derivatives. So, we deduce that

$$\|C_\delta^{-1/2}R_\delta\|_{\delta,2,q}\le L_q\rho_\delta$$

and the proof is completed. $\square$

We are now ready for the

Proof of Proposition 4.5. Consider the decomposition (4.8): we have $p_{X_T}(y)=p_F(0)$. We use Lemma 4.6 and Theorem 3.3. So, there exists δ_0 such that for $\delta<\delta_0$ if $|F_{T-\delta}|<\delta^{3/2}\rho_\delta$ then $|F_{T-\delta}|_\delta<c_2\rho_\delta$. We take now $\delta_1<\delta_0$ and $r=c_2\rho_{\delta_1}$. So, there exists $\delta<\delta_1$ such that $\theta_{\delta,q_d}=\|C_\delta^{-1/2}R_\delta\|_{\delta,2,q_d}\le a_de^{-r^2}$. Therefore, $\widetilde\Gamma_{\delta,r}(0)\supset\{|F_{T-\delta}|<\delta^{3/2}\rho_\delta\}$ and by the support theorem one has $\mathbb{P}(|F_{T-\delta}|<\delta^{3/2}\rho_\delta)=\mathbb{P}(|X_{T-\delta}-x_{T-\delta}(\phi)|<\delta^{3/2}\rho_\delta)>0$, so Theorem 3.3 allows one to conclude. $\square$

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